

# On the Frattini lemma

V. S. Monakhov

February 28, 2012

## Abstract

Let  $K$  be a subgroup of a finite group  $G$ , and suppose that  $G = KN_G(P)$  for every Sylow subgroup  $P$  of  $K$ . Then the subgroup  $K$  is normal in  $G$ .

**Keywords:** finite group, normal subgroup, Sylow subgroup.

MSC2010 20D10, 20D20, 20E34

The following assertion is often used in group theory.

**Lemma (Frattini).** *Let  $K$  be a normal subgroup of a finite group  $G$ , and suppose that  $P$  is a Sylow subgroup of  $K$ . Then  $G = KN_G(P)$ .*

It turns out that the converse of this assertion is also true.

**Lemma.** *Let  $K$  be a subgroup of a finite group  $G$ , and suppose that  $G = KN_G(P)$  for every Sylow subgroup  $P$  of  $K$ . Then the subgroup  $K$  is normal in  $G$ .*

**PROOF.** Let  $P_1, P_2, \dots, P_n$  be the set of all Sylow subgroups of  $K$ ,  $x_i \in P_i$ ,  $g \in G$ . Since by hypothesis,

$$G = KN_G(P_i) = N_G(P_i)K,$$

for all  $i$ , it follows that  $g = a_i b$ ,  $a_i \in N_G(P_i)$ ,  $b \in K$ . We have  $x_i^g = x_i^{a_i b} \in P_i^{a_i b} = P_i^b \subseteq K$ . Since  $\langle P_1, P_2, \dots, P_n \rangle = K$ , we see that every element of  $K$  can be written as a product of elements from  $P_1 \cup P_2 \cup \dots \cup P_n$ . If  $x \in K$  and  $g \in G$  are arbitrary elements, then

$$x = x_1 x_2 \dots x_n y_1 y_2 \dots y_n \dots z_1 z_2 \dots z_n, \quad x_i, y_i, z_i \in P_i,$$

$$x^g = (x_1 x_2 \dots x_n)^g = x_1^g x_2^g \dots x_n^g \in K,$$

which means that the subgroup  $K$  is normal in  $G$ . The lemma is proved.

V. S. MONAKHOV

Department of mathematics, Gomel F. Scorina State University, Gomel 246019, BELARUS

E-mail address: Victor.Monakhov@gmail.com